

## On nilpotent elements in a nearring of polynomials

EBRAHIM HASHEMI<sup>1,\*</sup>

<sup>1</sup> *Department of Mathematics, Shahrood University of Technology, P.O.Box  
316-3619995161, Shahrood, Iran*

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**Abstract.** For a ring  $R$ ,  $R[x]$  is a left nearring under addition and substitution, and we denote it by  $(R[x], +, \circ)$ . In this note, we show that if  $\text{nil}(R)$  is a locally nilpotent ideal of  $R$ , then  $\text{nil}(R[x], +, \circ) = \text{nil}(R)_0[x]$ , where  $\text{nil}(R)$  is the set of nilpotent elements of  $R$  and  $\text{nil}(R)_0[x]$  is the 0-symmetric left nearring of polynomials with coefficients in  $\text{nil}(R)$ . As a corollary, if  $R$  is a 2-primal ring, then  $\text{nil}(R[x], +, \circ) = \text{nil}(R)_0[x]$ .

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### 1. Introduction

Throughout this paper, all rings are associative and unitary and all nearrings are left nearrings; subrings of a ring need not have the same unit, and *subrng* will denote a subring without unit. For a ring or nearring  $N$ ,  $\text{nil}(N)$  denotes the set of nilpotent elements of  $N$ . Also,  $P(R)$  denotes the prime radical of a ring  $R$ . Recall that a ring or a nearring is said to be *reduced* if it has no nonzero nilpotent elements.

Rege and Chhawchharia [20] introduce the notion of an Armendariz ring. A ring  $R$  is called *Armendariz* if whenever  $f(x)g(x) = 0$  where  $f(x) = a_0 + a_1x + \cdots + a_nx^n$  and  $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ , then  $a_ib_j = 0$  for each  $i, j$ . The name of the ring was given to E. Armendariz who proved in [3] that reduced rings satisfied this condition. The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring  $R$  and the annihilators of the polynomial ring  $R[x]$ . Let us recall two known facts: A ring  $R$  is called *Baer* by Kaplansky [11] if the right annihilator of every nonempty subset of  $R$  is generated by an idempotent. An example of Chon shows that the matrix ring  $M_2(\mathbb{Z})$  is Baer but  $M_2(\mathbb{Z})[x]$  is not. A well-known example of Kerr [12] shows that there exists a right Goldie ring  $R$  such that  $R[x]$  is not right Goldie. But, for an Armendariz ring  $R$ ,  $R$  is Baer if and only if  $R[x]$  is Baer (Armendariz [3]; Kim and Lee [14]), and  $R$  is right Goldie if and only if  $R[x]$  is right Goldie (Hirano [8]). The reason behind these is a natural bijection between the set of annihilators of  $R$  and the set of annihilators of  $R[x]$  (see Hirano [8]). We refer to [1, 2, 3, 9, 10, 14, 15, 16, 17, 20] for more detail on Armendariz rings.

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\*Corresponding author. *Email address:* eb\_hashemi@yahoo.com (E. Hashemi)

Recall from [4]  $R$  is said to satisfy the IFP (*insertion of factors property*) if  $r_R(a) = \{b \in R \mid ab = 0\}$  is an ideal for all  $a \in R$ . Reduced rings satisfy the IFP. Shin [21] proved that  $R$  is a division ring if and only if  $R$  is a von Neumann regular prime ring and satisfies the IFP. Smoktunowicz [22] showed that there exists a nil ring  $R$  such that  $R[x]$  is not nil. But if  $R$  satisfies the IFP, then Liu and Zhao [16] proved that  $\text{nil}(R)$  is an ideal of  $R$  and  $\text{nil}(R[x]) = \text{nil}(R)[x]$ . Also Antoine [2] proved that if  $R$  is an Armendariz ring, then  $\text{nil}(R)$  is a subrng of  $R$  and  $\text{nil}(R[x]) = \text{nil}(R)[x]$ . Properties, examples and counterexamples of rings which satisfy the IFP are given in [4, 7, 8, 10, 18, 21, 23].

A ring  $R$  is called *2-primal* if the prime radical of  $R$  coincides with the set of all nilpotent elements of  $R$  (see [6] for details). The class of 2-primal rings is also closed under subrings by [6, Proposition 2.2]. If  $R$  satisfies the IFP, then  $R$  is 2-primal.

Let  $R$  be a ring. Since  $R[x]$  is an abelian nearring under addition and substitution, it is natural to investigate the nearring of polynomials  $(R[x], +, \circ)$ . The binary operation of substitution, denoted by  $\circ$ , of one polynomial into another is both natural and important in the theory of polynomials. We adopt the convention that for polynomials  $(x)g$  and  $(x)f = \sum_{i=0}^m f_i x^i \in R[x]$ ,  $(x)g \circ (x)f = \sum_{i=0}^m f_i ((x)g)^i$ . For example,  $(a_0 + a_1 x) \circ x^2 = (a_0 + a_1 x)^2 = a_0^2 + (a_0 a_1 + a_1 a_0)x + a_1^2 x^2$ . However, the operation " $\circ$ ", left distributes but does not right distribute over addition. Thus  $(R[x], +, \circ)$  forms a left nearring but not a ring. Unless specifically indicated otherwise,  $R[x]$  denotes the left nearring of polynomials  $(R[x], +, \circ)$  with coefficients from  $R$  and  $R_0[x] = \{f \in R[x] \mid f \text{ has zero constant term}\}$  is the 0-symmetric left nearring of polynomials with coefficients in  $R$ .

We say that a set  $S \subseteq R$  is *locally nilpotent* if for any subset  $\{s_1, s_2, \dots, s_n\} \subseteq S$ , there exists an integer  $t$ , such that any product of  $t$  elements from  $\{s_1, s_2, \dots, s_n\}$  is zero.

Antoine [2, Corollaries 3.3 and 5.2] proved that if  $R$  is an Armendariz ring, then  $\text{nil}(R)$  is a subrng of  $R$  and  $\text{nil}(R)[x] = \text{nil}(R[x])$ . Hence  $\text{nil}(R)$  is a locally nilpotent subrng of  $R$ , when  $R$  is an Armendariz ring.

The following examples show that there exist non Armendariz rings such that the set of its nilpotent elements is a locally nilpotent ideal.

**Example 1.** Let  $\mathbb{Z}$  be the ring of integers and let

$$R = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a - b \equiv c \equiv 0 \pmod{2} \right\}.$$

Then by [14, Example 13],  $R$  is not Armendariz. Since

$$\text{nil}(R) = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \equiv 0 \pmod{2} \right\},$$

hence  $\text{nil}(R)$  is a locally nilpotent ideal of  $R$ .

**Example 2.** Let  $T$  be a reduced ring and

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in T \right\}.$$

Let

$$S = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \mid A, B \in R \right\}.$$

Then by [14, Example 5],  $S$  is not Armendariz. Since

$$\text{nil}(S) = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \mid A \in \text{nil}(R) \text{ and } B \in R \right\},$$

hence  $\text{nil}(S)$  is a locally nilpotent ideal of  $S$ .

If  $R$  satisfies the IFP, then  $R$  is *abelian* (i.e., each idempotent of  $R$  is central). The following example shows that there exists a ring  $R$  such that it does not satisfy the IFP, but  $\text{nil}(R)$  is a locally nilpotent ideal of  $R$ .

**Example 3.** Let  $F$  be a division ring and consider the 2-by-2 upper triangular ring

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}.$$

Then  $R$  does not satisfy the IFP, since  $R$  is not abelian. But

$$\text{nil}(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F \right\}$$

is a locally nilpotent ideal of  $R$ .

## 2. Nilpotent elements in a nearring of polynomials

**Definition 1.** Let  $R$  be a ring. We say  $R$  has property  $(*)$ , whenever  $(x)f = a_0 + a_1x + \cdots + a_mx^m$ ,  $(x)g = b_0 + b_1x + \cdots + b_nx^n$  are elements of nearring  $(R[x], +, \circ)$  and  $f \circ g \in \text{nil}(R)[x]$ , then  $a_ib_j \in \text{nil}(R)$  for  $i = 1, \dots, m$ ,  $j = 0, 1, \dots, n$ .

By [5, Lemma 3.4], every reduced ring has property  $(*)$ .

**Proposition 1.** Let  $I$  be a nil ideal of a ring  $R$ . Then  $R/I$  has property  $(*)$  if and only if  $R$  has property  $(*)$ .

**Proof.** We denote  $\bar{R} = R/I$ . Since  $I$  is nil, then  $\text{nil}(\bar{R}) = \overline{\text{nil}(R)}$ . Let  $(x)f = \sum_{i=0}^m a_ix^i$  and  $(x)g = \sum_{j=0}^n b_jx^j$  be elements of nearring  $R[x]$ . Then  $f \circ g \in \text{nil}(R)[x]$ , if and only if  $(\sum_{i=0}^m \bar{a}_ix^i) \circ (\sum_{j=0}^n \bar{b}_jx^j) \in \text{nil}(\bar{R})[x]$ . Also,  $a_ib_j \in \text{nil}(R)$  if and only if  $\bar{a}_i\bar{b}_j \in \text{nil}(\bar{R})$ , for  $i = 1, \dots, m$  and  $j = 0, 1, \dots, n$ .  $\square$

**Proposition 2.** Let  $R$  be a ring and  $e$  a central idempotent element of  $R$ . Then the following statements are equivalent:

1.  $R$  has property  $(*)$ .
2.  $eR$  and  $(1 - e)R$  have property  $(*)$ .

**Proof.** (1)  $\Rightarrow$  (2) It is clear, since  $eR$  and  $(1-e)R$  are subrings of  $R$ .

(2)  $\Rightarrow$  (1) Let  $(x)f = \sum_{i=0}^m a_i x^i$  and  $(x)g = \sum_{j=0}^n b_j x^j$  be elements of nearring  $R[x]$  such that  $f \circ g \in \text{nil}(R)[x]$ . Let  $(x)f_1 = \sum_{i=0}^m e a_i x^i$ ,  $(x)f_2 = \sum_{i=0}^m (1-e) a_i x^i$ ,  $(x)g_1 = \sum_{j=0}^n e b_j x^j$  and  $(x)g_2 = \sum_{j=0}^n (1-e) b_j x^j$ . Then  $f_1 \circ g_1 = (\sum_{i=0}^m e a_i x^i) \circ (\sum_{j=0}^n e b_j x^j) = e.(f \circ g) \in \text{nil}(eR)[x]$  and  $f_2 \circ g_2 = (\sum_{i=0}^m (1-e) a_i x^i) \circ (\sum_{j=0}^n (1-e) b_j x^j) = (1-e).(f \circ g) \in \text{nil}((1-e)R)[x]$ , since  $f \circ g \in \text{nil}(R)[x]$  and  $e, (1-e)$  are central idempotent elements of  $R$ . Hence  $e a_i b_j$  and  $(1-e) a_i b_j$  are nilpotent, for each  $i = 1, \dots, m$  and  $j = 0, 1, \dots, n$ , since  $eR$  and  $(1-e)R$  have property (\*). Thus there exists  $t \geq 2$  such that  $(e a_i b_j)^t = ((1-e) a_i b_j)^t = 0$  for each  $i = 1, \dots, m$  and  $j = 0, 1, \dots, n$ . Hence  $(a_i b_j)^t = 0$  for each  $i = 1, \dots, m$  and  $j = 0, 1, \dots, n$ . Therefore  $R$  has property (\*).  $\square$

**Proposition 3.** Let  $R$  be a finite subdirect sum of rings which have property (\*). Then  $R$  has property (\*).

**Proof.** Let  $I_k$  ( $k = 1, 2, \dots, \ell$ ) be ideals of  $R$  such that  $R/I_k$  has property (\*) and  $\cap_{k=1}^{\ell} I_k = 0$ . Suppose that  $(x)f = \sum_{i=0}^m a_i x^i$  and  $(x)g = \sum_{j=0}^n b_j x^j$  be elements of nearring  $R[x]$  such that  $f \circ g \in \text{nil}(R)[x]$ . Then there exists  $p_{ij} \geq 1$ , such that  $(\bar{a}_i \bar{b}_j)^{p_{ij}} = 0$  in  $R/I_k$ . Thus  $(a_i b_j)^{p_{ij}} \in I_k$ . Set  $p = \max\{p_{ij} | i, j \geq 1\}$ . Then  $(a_i b_j)^{p_{ij}} \in I_k$ , for any  $k$ , which implies that  $(a_i b_j)^p = 0$ . Therefore  $R$  has property (\*).  $\square$

For a ring  $R$ , we denote the  $n$ -by- $n$  upper triangular and full matrix ring over  $R$  by  $T_n(R)$  and  $M_n(R)$ , respectively.

**Proposition 4.** A ring  $R$  has property (\*) if and only if, for any  $n$ ,  $T_n(R)$  has property (\*).

**Proof.** If  $T_n(R)$  has property (\*), then so  $R$  has property (\*) as a subring of  $T_n(R)$ .

Conversely, let  $(x)f = \sum_{i=0}^p A_i x^i$  and  $(x)g = \sum_{j=0}^q B_j x^j$  be elements of nearring  $T_n(R)[x]$  such that  $f \circ g \in \text{nil}(T_n(R))[x]$ . Let

$$A_i = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{pmatrix}$$

and

$$B_j = \begin{pmatrix} b_{11}^j & b_{12}^j & \cdots & b_{1n}^j \\ 0 & b_{22}^j & \cdots & b_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}^j \end{pmatrix}.$$

Then from  $f \circ g \in \text{nil}(T_n(R))[x]$  it follows that  $(\sum_{i=0}^p a_{ss}^i x^i) \circ (\sum_{j=0}^q b_{ss}^j x^j) \in \text{nil}(R)[x]$  for  $s = 1, \dots, n$ . Since  $R$  has property (\*),  $a_{ss}^i b_{ss}^j \in \text{nil}(R)$ , for each  $i = 1, \dots, p$ ,  $j = 0, 1, \dots, q$  and  $s = 1, \dots, n$ . Then  $A_i B_j \in \text{nil}(T_n(R))$  for each  $i = 1, \dots, p$ ,  $j = 0, 1, \dots, q$ . Therefore  $T_n(R)$  has property (\*).  $\square$

Let  $R$  be a ring. Then

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

is a subring of  $T_n(R)$ , for each  $n \geq 2$ . By a similar argument as used in the proof of Proposition 4, we can show that  $R$  has property (\*) if and only if, for any  $n$ ,  $R_n$  has property (\*).

The same idea can be used to prove the following.

**Proposition 5.** *Let  $R, S$  be rings and  ${}_R M_S$  an  $(R, S)$ -bimodule. Then  $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$  has property (\*) if and only if  $R$  and  $S$  have property (\*).*

**Theorem 1.** *If  $\text{nil}(R)$  is an ideal of  $R$ , then  $R$  has property (\*).*

**Proof.** Since  $R/\text{nil}(R)$  is a reduced ring, hence by [5, Lemma 3.4],  $R/\text{nil}(R)$  has property (\*). Hence by Proposition 1,  $R$  has property (\*).  $\square$

**Lemma 1** (see [16]). *If  $R$  satisfies the IFP, then*

1.  $\text{nil}(R)$  is a locally nilpotent ideal of  $R$ ;
2.  $\text{nil}(R[x]) = \text{nil}(R)[x]$ .

**Proposition 6.** *If  $R$  satisfies the IFP, then  $R$  has property (\*).*

**Proof.** It follows from Lemma 1 and Theorem 1.  $\square$

The following example shows that the condition “ $\text{nil}(R)$  be an ideal of  $R$ ” in Theorem 1 is not superfluous.

**Example 4.** *Let  $F$  be a field and  $S = M_2(F)$ . Then  $\text{nil}(S)$  is not ideal of  $R$ . Let*

$$({}_x)f = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} x$$

and

$$({}_x)g = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^2$$

be polynomials in  $S[x]$ . Then  $f \circ g = 0 \in \text{nil}(S)[x]$ , but

$$\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin \text{nil}(S).$$

**Lemma 2.** *Let  $\text{nil}(R)$  be an ideal of  $R$ , and  $a_1, a_2, \dots, a_n, a, b \in R$ .*

1. *If  $ab \in \text{nil}(R)$ , then  $arb \in \text{nil}(R)$  for each  $r \in R$ .*

2. If  $ab^n \in \text{nil}(R)$  for some  $n \geq 1$ , then  $ab \in \text{nil}(R)$ .

3. If  $b_1 b_2 \cdots b_m \in \text{nil}(R)$ , where  $b_i \in \{a_1, a_2, \dots, a_n\}$ , then  $a_1 a_2 \cdots a_n \in \text{nil}(R)$ .

**Proof.** The details are left to the reader.  $\square$

For any  $(x)f \in R[x]$ , we denote by  $C_f$  the set of all coefficients of  $f$ . Let  $C_f^* = C_f - \{a_0\}$ , where  $a_0$  is the constant term of  $f$ .

**Proposition 7.** Let  $(x)f_1, (x)f_2, \dots, (x)f_n$  be elements of nearring  $R[x]$ , such that  $f_1 \circ f_2 \circ \cdots \circ f_n \in \text{nil}(R)[x]$ . If  $\text{nil}(R)$  is an ideal of  $R$ , then  $C_{f_1}^* C_{f_2}^* \cdots C_{f_n}^* \subseteq \text{nil}(R)$ .

**Proof.** We use induction on  $n$ . The case  $n = 2$  follows from Theorem 1.

Suppose  $n > 2$ . Consider  $g = f_2 \circ f_3 \circ \cdots \circ f_n$ . Then  $f_1 \circ g \in \text{nil}(R)[x]$  and hence by Theorem 1,  $a_1 a_g \in \text{nil}(R)$  where  $a_g \in C_g$  and  $a_1 \in C_{f_1}^*$ . Therefore for all  $a_1 \in C_{f_1}^*$ ,

$$\begin{aligned} g \circ a_1 x &= (f_2 \circ f_3 \circ \cdots \circ f_n) \circ a_1 x = f_2 \circ f_3 \circ \cdots \circ f_{n-1} \circ (f_n \circ a_1 x) \\ &= f_2 \circ f_3 \circ \cdots \circ f_{n-1} \circ (a_1 f_n) \in \text{nil}(R)[x] \end{aligned}$$

and by induction, since the coefficients of  $a_1 f_n$  are  $a_1 a_n$ , where  $a_n$  is a coefficient of  $f_n$ , we obtain  $a_2 a_3 \cdots a_{n-1} a_1 a_n \in \text{nil}(R)$ . Hence  $C_{f_1}^* C_{f_2}^* \cdots C_{f_n}^* \subseteq \text{nil}(R)$ , by Lemma 2.  $\square$

**Theorem 2.** Let  $(x)f = a_0 + a_1 x + \cdots + a_m x^m$  be a nilpotent element of nearring  $R[x]$  and  $\text{nil}(R)$  an ideal of  $R$ . Then  $a_i \in \text{nil}(R)$  for  $i = 0, 1, \dots, m$ .

**Proof.** Let  $(x)f = a_0 + a_1 x + \cdots + a_m x^m \in \text{nil}(R[x])$ . Then there exists  $k \geq 2$  such that

$$f^k = \underbrace{f \circ f \circ f \cdots \circ f}_k = 0 \in \text{nil}(R)[x].$$

By Proposition 7,  $a_i \in \text{nil}(R)$  for each  $i = 1, \dots, m$ . We claim that  $a_0 \in \text{nil}(R)$ . The constant term of  $f^k$  is  $a_0 + \beta$ , where  $\beta$  is a sum of elements  $a_{i_1} a_{i_2} \cdots a_{i_t}$  such that  $t \geq 2$  and  $\{a_{i_1}, a_{i_2}, \dots, a_{i_t}\} \cap \{a_1, a_2, \dots, a_m\} \neq \emptyset$ . Then  $\beta \in \text{nil}(R)$ , and since  $a_0 + \beta \in \text{nil}(R)$ , we have  $a_0 \in \text{nil}(R)$ . Therefore  $a_i \in \text{nil}(R)$ , for  $i = 0, 1, \dots, m$ .  $\square$

**Theorem 3.** If  $\text{nil}(R)$  is a locally nilpotent ideal of a ring  $R$ , then  $\text{nil}(R[x]) = \text{nil}(R)_0[x]$ .

**Proof.** Let  $(x)f = a_0 + a_1 x + \cdots + a_m x^m$  be a nilpotent element of nearring  $R[x]$ . By Theorem 2,  $a_i \in \text{nil}(R)$  for  $i = 0, 1, \dots, m$ . Thus  $\{a_0, a_1, \dots, a_m\} \subseteq \text{nil}(R)$ , and since  $\text{nil}(R)$  is a locally nilpotent subset of  $R$ , there exists  $t \geq 2$  such that  $\{a_0, a_1, \dots, a_m\}^t = 0$ . Since  $f \in \text{nil}(R[x])$ , hence

$$f^k = \underbrace{f \circ f \circ f \cdots \circ f}_k = 0,$$

for some  $k \geq t$ . For each  $j \geq 1$ , the coefficient of  $x^j$  in the polynomial  $f^k$  is a sum of elements  $a_{i_1} a_{i_2} \cdots a_{i_\ell}$ , where  $a_{i_r} \in \{a_0, a_1, \dots, a_m\}$ , and  $\ell \geq k$ . Also the

constant term of the polynomial  $f^k$  is  $a_0 + a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-2}a_0 + \alpha$ , where  $\alpha$  is a sum of elements  $a_{i_1}a_{i_2}\cdots a_{i_\ell}$ , where  $a_{i_r} \in \{a_0, a_1, \dots, a_m\}$  and  $\ell \geq k$ . Since  $\{a_0, a_1, \dots, a_m\}^t = 0$ , hence  $\alpha = 0$ , and since  $a_0 + a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-2}a_0 + \alpha = 0$ , we have  $a_0 + a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-2}a_0 = 0$ . Multiplying this equation by  $a_1$  from the left yields  $a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-1}a_0 = 0$ , and since  $a_1^{k-1}a_0 = 0$  we have  $a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-2}a_0 = 0$ . Hence  $a_0 = 0$  and  $\text{nil}(R[x]) \subseteq \text{nil}(R)_0[x]$ .

Now let  $(x)f = a_1x + \cdots + a_mx^m \in \text{nil}(R)_0[x]$ . Since  $\text{nil}(R)$  is a locally nilpotent subset of  $R$ , there exists  $t \geq 2$  such that  $\{a_1, \dots, a_m\}^t = 0$ . Since for each  $j \geq 2$ , the coefficient of  $x^j$  in the polynomial

$$f^t = \underbrace{f \circ f \circ f \cdots \circ f}_t$$

is a sum of elements  $a_{i_1}a_{i_2}\cdots a_{i_\ell}$ , where  $a_{i_r} \in \{a_1, \dots, a_m\}$  and  $\ell \geq t$ , hence  $f^t = 0$ . Therefore  $\text{nil}(R)_0[x] \subseteq \text{nil}(R[x])$  and hence  $\text{nil}(R[x]) = \text{nil}(R)_0[x]$ .  $\square$

By [13, Proposition 10.31], the sum of all locally nilpotent ideals in a ring  $R$  (denoted by  $L\text{-rad } R$ ) is locally nilpotent, and  $P(R) \subseteq L\text{-rad } R \subseteq \text{nil}(R)$ . Then  $P(R) = L\text{-rad } R = \text{nil}(R)$ , if  $R$  is a 2-primal ring. Thus we have the following result:

**Corollary 1.** *If  $R$  is a 2-primal ring, then  $\text{nil}(R[x]) = \text{nil}(R)_0[x]$ .*

**Corollary 2.** *If  $R$  satisfies the IFP, then  $\text{nil}(R[x]) = \text{nil}(R)_0[x]$ .*

**Corollary 3.** *If  $R$  is an Armendariz ring and  $\text{nil}(R)$  an ideal of  $R$ , then  $\text{nil}(R[x]) = \text{nil}(R)_0[x]$ .*

**Proof.** Since  $R$  is an Armendariz ring, hence by [2, Corollary 5.2]  $\text{nil}(R)[x] = \text{nil}(R[x])$ . Thus by [1, Proposition 1],  $\text{nil}(R)$  is a locally nilpotent subset of  $R$ . Now the result follows from Theorem 3.  $\square$

**Proposition 8.** *If  $\text{nil}(R)$  is a locally nilpotent ideal of  $R$ , then  $\text{nil}(R[x])$  is a right ideal of  $(R[x], +, \circ)$ .*

**Proof.** Let

$$\begin{aligned} (x)f &= f_0 + f_1x + \cdots + f_mx^m, \\ (x)h &= h_0 + h_1x + \cdots + h_mx^m \in R[x] \end{aligned}$$

and

$$(x)g = g_1x + \cdots + g_mx^m \in \text{nil}(R[x]).$$

Then

$$(f+g) \circ h - f \circ h = h_1[(f+g) - f] + h_2[(f+g)^2 - f^2] + \cdots + h_m[(f+g)^m - f^m].$$

Since for each  $i \geq 2$ ,  $[(f+g)^i - f^i] \in \text{nil}(R)_0[x]$ , hence  $(f+g) \circ h - f \circ h \in \text{nil}(R)_0[x]$ . Thus  $\text{nil}(R[x])$  is a right ideal of  $(R[x], +, \circ)$ .  $\square$

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